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Existence and Exponential Decay for a Kirchhoff–Carrier Model with Viscosity

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In this work we study the existence of global solutions and exponential decay for the following nonlinear problem:

$$\left|\begin{array}{l} \dfrac{\partial^2 y}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla y|^2\,dx\right)\Delta y - \dfrac{\partial}{\partial t}\Delta y = f \qquad \text{in } Q = \Omega \times (0,\infty), \\ y = 0 \qquad \text{on } \Sigma_1 = \Gamma_1 \times (0,\infty), \\ M\left(\displaystyle\int_\Omega |\nabla y|^2\,dx\right)\dfrac{\partial y}{\partial \nu} + \dfrac{\partial}{\partial t}\left(\dfrac{\partial y}{\partial \nu}\right) = g \qquad \text{on } \Sigma_0 = \Gamma_0 \times (0,\infty), \\ y(0) = y^0, \qquad \dfrac{\partial y}{\partial t}(0) = y^1 \qquad \text{in } \Omega, \end{array}\right. \qquad (*)$$

where M is a C^1 function, $M(\lambda) \geq \lambda_0 > 0$; $\forall \lambda \geq 0$. © 1998 Academic Press

Key Words: Kirchhoff–Carrier; Galerkin method; exponential decay; viscosity.

1. INTRODUCTION

Let Ω be a bounded domain of $\mathbf{R}^n$ with C^2 boundary Γ. Let (Γ_0, Γ_1) be a partition of Γ, both parts with positive measure and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \varnothing$ (this assumption excludes the simply connected regions). Let ν be the unit normal vector pointing toward the exterior of Ω and let $\partial/\partial\nu$ be the normal derivative.

Let $M \in C^1([0,\infty), \mathbf{R})$ be a function such that

$$M(\lambda) \geq \lambda_0 > 0, \qquad \forall \lambda \geq 0. \tag{1.1}$$

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In this paper we prove existence of strong and weak solutions of $(*)$ as well as the uniform decay, where the latter is obtained assuming that appropriate hypotheses on f and g hold.

Our model was inspired in the work of J. L. Lions [9], where the following problem was considered:

$$\left|\begin{array}{l} y_{tt} - M\left(|\nabla y|^2\right) \Delta y = f \qquad \text{in } \Omega \times (0,T), \\ y = 0 \qquad \text{on } \Gamma \times (0,T), \\ y(0) = y^0, \qquad y'(0) = y^1 \qquad \text{in } \Omega. \end{array}\right.$$

This problem has its origin in the canonical model of Kirchhoff and Carrier which describes small vibrations of an elastic stretched string. More precisely we have

$$\rho \frac{\partial^2 y}{\partial t^2} = \left(P_0 + \frac{Eh}{2L} \int_0^L \left| \frac{\partial y}{\partial x}(x,t) \right|^2 dx \right) \frac{\partial^2 y}{\partial x^2}, \qquad 0 \leq x \leq L, t \geq 0,$$

where y is the lateral deflection, ρ is the mass density, h is the cross-sectional area, L is the length, E is Young's modulus, and P_0 is the initial axial tension.

The bibliography of works in this direction is truly long. Among the classical works we can cite, for instance, Menzala [12], Arosio and Spagnolo [1], Rivera [14], and Ebihara *et al.* [3]. Now, related with damped problems, we refer the reader to the works of Yamada [16], Vasconcellos and Teixeira [15], and Medeiros and Milla Miranda [11]. The latter studied the following hyperbolic problem:

$$\left|\begin{array}{l} y_{tt} - M\left(|\nabla y|^2\right) \Delta y + (-\Delta)^{\alpha} y_t = f \qquad \text{in } \Omega \times (0,T), \\ y = 0 \qquad \text{on } \Gamma \times (0,T), \\ y(0) = y^0, \qquad y'(0) = y^1 \qquad \text{in } \Omega, \end{array}\right.$$

where the global solution was obtained and the exponential decay was proved when $f = 0$.

According to our best knowledge the several works related to those kinds of problems treat homogeneous boundary conditions, and in order to obtain the existence of solutions the authors employ Galerkin's method and make use of a "special basis," that is, ones formed by eigenfunctions $(\omega_j)_{j \in \mathbf{N}}$ which satisfy the property

$$-\Delta \omega_j = \lambda_j \omega_j. \tag{1.2}$$

In this work we also use Galerkin's approximation and taking into account the nonhomogeneous boundary condition, we cannot use the basis satisfying (1.2). Hence, we are not able to pass to the limit using standard arguments of compacity and as a consequence we have to find other arguments which allow us to do it. This is the aim of our work.

It is convenient to observe that when $M = 1$ an asymptotic regularization procedure was proved by Hsiao and Sprekels [5]. Problems without viscosity, that is, when $\Delta y' = 0$, and such that $M = 1$ with the damping occurring in the boundary were studied by many authors; see Quinn and Russel [13], Chen [2], Lagnese [7], Lasiecka and Tataru [8], and Komornik and Zuazua [6]. The latter also considered the case when $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, using the same arguments as Grisvard [4].

To obtain exponential decay we used the perturbed energy method; see, for example, Komornik and Zuazua [6]. Stability problems with nonhomogeneous conditions require a special treatment since we do not have any information about the influence of the inner products $(f(t), y'(t))_{L^2(\Omega)}$ and $(g(t), y'(t))_{\Gamma_0}$ on the energy

$$E(t) = \frac{1}{2}\left[|y'(t)|^2 + \hat{M}\left(|\nabla y(t)|^2\right)\right] \tag{1.3}$$

or about the sign of the derivative $E'(t)$. Here

$$\hat{M}(\lambda) = \int_0^t M(s)\, ds. \tag{1.4}$$

At this point it is important to observe that it is not possible to obtain exponential decay when we consider degenerate problems, that is, those that satisfy the condition $M(0) = 0$. However, when $M(\lambda) \geq \lambda_0 > 0$, Zuazua [17] proved that exponential decay holds for a general class of damped hyperbolic problems.

Our paper is organized as follows. In Section 2 we give the notation and state the main result. In Section 3 we study the existence and uniqueness of strong and weak solutions while in Section 4 we obtain exponential decay for solutions obtained in Section 3.

2. NOTATION AND MAIN RESULT

In this section we present some notation that will be used throughout this paper and we state the main result.

Let

$$V = \left\{v \in H^1(\Omega);\, v = 0 \text{ on } \Gamma_1\right\}$$

endowed with the topology given by the norm $|\nabla \cdot|_{L^2(\Omega)}$. V is a Hilbert subspace of $H^1(\Omega)$.

We write

$$(u,v) = \int_\Omega u(x)v(x)\,dx, \qquad (u,v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\,d\Gamma.$$

Now we are in a condition to state our main result.

THEOREM 2.1. *Let*

$$\{y^0, y^1, f, g\} \in V \times L^2(\Omega) \times L^2\big(0,\infty; L^2(\Omega)\big) \times L^2\big(0,\infty; L^2(\Gamma_0)\big).$$

Then problem $(*)$ *possesses a unique weak solution such that*

$$y \in C^0\big([0,\infty); V\big) \cap C^1\big([0,\infty); L^2(\Omega)\big), \tag{2.1}$$

$$y' \in L^2(0,\infty; V). \tag{2.2}$$

Moreover, assuming that for a large t the inequality

$$\int_0^t \exp\left(\frac{\varepsilon}{2} C_2 s\right)\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right) ds \le \alpha t^\beta \tag{2.3}$$

holds for some positive constants ε, C_2, α, *and* β *we obtain the following energy decay*:

$$E(t) \le C \exp\left(-\frac{\varepsilon}{2} C_2 t\right), \qquad \forall t \ge 0 \text{ and } \forall \varepsilon \in (0, \varepsilon_0],$$

where C *and* ε_0 *are positive constants.*

Remark 1. Hypothesis (2.3) means that the map

$$t \to \int_0^t \exp\left(\frac{\varepsilon}{2} C_2 s\right)\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right) ds$$

is bounded by a polynomial $P(t)$.

3. EXISTENCE AND UNIQUENESS OF STRONG AND WEAK SOLUTIONS

In order to obtain strong solutions let us consider

$$\{y^0, y^1, f, g\} \in V \cap H^2(\Omega) \times V \cap H^2(\Omega) \times L^2\big(0,\infty; L^2(\Omega)\big) \times H^1\big(0,\infty; L^2(\Gamma_0)\big). \tag{3.1}$$

The variational formulation associated with problem $(*)$ is given by

$$\begin{aligned}(y''(t),w) + M\left(|\nabla y(t)|^2\right)(\nabla y(t),\nabla w) + (\nabla y'(t),\nabla w)\\ = (f(t),w) + (g(t),w)_{\Gamma_0}.\end{aligned} \tag{3.2}$$

We represent by $(\omega_j)_{j \in \mathbf{N}}$ a basis in $V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and by V_m the subspace of $V \cap H^2(\Omega)$ generated by the m-first vectors $\omega_1, \dots, \omega_m$, and we define

$$y_m(t) = \sum_{i=1}^{m} g_{im}(t)\,\omega_i, \tag{3.3}$$

where $v_m(t)$ is the solution of the following Cauchy problem:

$$\begin{aligned}\left(y_m''(t), \omega_j\right) + M\left(|\nabla y_m(t)|^2\right)\left(\nabla y_m(t), \nabla\omega_j\right) + \left(\nabla y_m'(t), \nabla\omega_j\right)\\ = \left(f(t), \omega_j\right) + \left(g(t), \omega_j\right)_{\Gamma_0}\end{aligned} \tag{3.4}$$

with initial data

$$\begin{aligned} y_m(0) &= y_{0m} \to y^0 \qquad \text{in } V \cap H^2(\Omega)\\ y_m'(0) &= y_{1m} \to y^1 \qquad \text{in } V \cap H^2(\Omega),\end{aligned} \tag{3.5}$$

The approximate system is a system of m ordinary differential equations. It is easy to see that (3.4) has a local solution in $[0, t_m)$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the first estimate we are going to obtain below.

A Priori Estimates

The First Estimate

Multiplying (3.4) by $g'_{jm}(t)$ and summing over j from (1.4) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{|y_m'(t)|^2 + \hat{M}\left(|\nabla y_m(t)|^2\right)\right\} + |\nabla y_m'(t)|^2\\ = \left(f(t), y_m'(t)\right) + \left(g(t), y_m'(t)\right)_{\Gamma_0}.\end{aligned} \tag{3.6}$$

Let C_0 be a positive constant such that

$$|v|_{\Gamma_0} \leq C_0 |\nabla v|; \qquad \forall v \in V,$$

and let us consider an arbitrary $\eta > 0$. Then, from (3.6) we obtain

$$\frac{1}{2}\frac{d}{dt}\Big\{|y_m'(t)|^2 + \hat{M}\big(|\nabla y_m(t)|^2\big)\Big\} + (1-\eta)|\nabla y_m'(t)|^2$$
$$\leq \frac{1}{2}|f(t)|^2 + \frac{1}{2}|y_m'(t)|^2 + \frac{C_0^2}{4\eta}|g(t)|_{\Gamma_0}^2. \tag{3.7}$$

Integrating (3.7) over $(0, t)$ and taking (1.1) into account we get

$$\frac{1}{2}|y_m'(t)|^2 + \frac{\lambda_0}{2}|\nabla y_m(t)|^2 + (1-\eta)\int_0^t |\nabla y_m'(s)|^2\, ds$$
$$\leq \frac{1}{2}|y_{1m}|^2 + \frac{1}{2}\hat{M}\big(|\nabla y_{0m}|^2\big) + \|f\|^2_{L^2(0,T;L^2(\Omega))} + \frac{C_0^2}{4\eta}\|g\|^2_{L^2(0,t;L^2(\Gamma_0))}$$
$$+ \frac{1}{2}\int_0^t |y_m'(s)|^2\, ds. \tag{3.8}$$

From (3.5) and (3.8), choosing $\eta > 0$ small enough and employing Gronwall's lemma we obtain the first estimate

$$|y_m'(t)|^2 + |\nabla y_m(t)|^2 + \int_0^t |\nabla y_m'(s)|^2\, ds \leq L_1, \tag{3.9}$$

where L_1 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0, T]$.

The Second Estimate

Let $m_2 \geq m_1$ be two natural numbers and consider $z_m = y_{m_2} - y_{m_1}$. Then, from (3.4) we can write

$$\frac{d}{dt}|z_m'(t)|^2 + 2|\nabla z_m'(t)|^2$$
$$= -2M\big(|\nabla y_{m_2}(t)|^2\big)\big(\nabla y_{m_2}(t), \nabla z_m'(t)\big)$$
$$+ 2M\big(|\nabla y_{m_1}(t)|^2\big)\big(\nabla y_{m_1}(t), \nabla z_m'(t)\big). \tag{3.10}$$

On the other hand, we note that

$$\frac{d}{dt}\Big\{M\big(|\nabla y_{m_2}(t)|^2\big)|\nabla z_m(t)|^2\Big\}$$
$$= 2M'\big(|\nabla y_{m_2}(t)|^2\big)\big(\nabla y_{m_2}(t), \nabla y_{m_2}'(t)\big)|\nabla z_m(t)|^2$$
$$+ 2M\big(|\nabla y_{m_2}(t)|^2\big)\big(\nabla z_m(t), \nabla z_m'(t)\big). \tag{3.11}$$

Then, from (3.10) and (3.11) adding and subtracting appropriate terms it follows that

$$\frac{d}{dt}\left\{|z_m'(t)|^2 + M\left(|\nabla y_{m_2}(t)|^2\right)|\nabla z_m(t)|^2\right\} + 2|\nabla z_m'(t)|^2$$
$$= 2\left(M\left(|\nabla y_{m_1}|^2\right) - M\left(|\nabla y_{m_2}|^2\right)\right)\left(\nabla y_{m_1}(t), \nabla z_m'(t)\right)$$
$$+ 2M'\left(|\nabla y_{m_2}(t)|^2\right)\left(\nabla y_{m_2}(t), \nabla y_{m_2}'(t)\right)|\nabla z_m(t)|^2. \quad (3.12)$$

We note that from the first estimate (3.9) we have

$$\left|M\left(|\nabla y_{m_1}(t)|^2\right) - M\left(|\nabla y_{m_2}(t)|^2\right)\right|$$
$$\leq \int_{|\nabla y_{m_1}|^2}^{|\nabla y_{m_2}|^2} |M'(\xi)|\, d\xi$$
$$\leq k_0\left||\nabla y_{m_2}(t)|^2 - |\nabla y_{m_1}(t)|^2\right|$$
$$\leq k_0\left[|\nabla y_{m_2}(t)| + |\nabla y_{m_1}(t)|\right]|\nabla y_{m_2}(t) - \nabla y_{m_1}(t)|$$
$$\leq k_1|\nabla z_m(t)|, \quad (3.13)$$

where k_0 and k_1 are positive constants.

From (3.13), taking (3.9) into account and considering an arbitrary $\eta > 0$, we get

$$2\left(M\left(|\nabla y_{m_1}|^2\right) - M\left(|\nabla y_{m_2}|^2\right)\right)\left(\nabla y_{m_1}(t), \nabla z'(t)\right)$$
$$\leq 2k_1|\nabla z_m(t)|\,|\nabla y_{m_1}(t)|\,|\nabla z_m'(t)|$$
$$\leq \frac{k_2}{4\eta}|\nabla z_m(t)|^2 + \eta|\nabla z_m'(t)|^2, \quad (3.14)$$

where k_2 is a positive constant.

Again, from the first estimate we deduce the existence of a positive constant k_3 such that

$$2M'\left(|\nabla y_{m_2}(t)|^2\right)\left(\nabla y_{m_2}(t), \nabla y_{m_2}'(t)\right)|\nabla z_m(t)|^2$$
$$\leq k_3\left|\nabla y_{m_2}'(t)\right|\,|\nabla z_m(t)|^2. \quad (3.15)$$

Combining (3.12), (3.14), and (3.15) it follows that

$$\begin{aligned}
&\frac{d}{dt}\Big\{|z_m'(t)|^2 + M\Big(|\nabla y_{m_2}(t)|^2\Big)|\nabla z_m(t)|^2\Big\} + (2-\eta)|\nabla z_m'(t)|^2 \\
&\quad \le \frac{k_2}{4\eta}|\nabla z_m(t)|^2 + k_3\big|\nabla y_{m_2}'(t)\big|\,|\nabla z_m(t)|^2 \\
&\quad \le k_4(\eta)\Big(1 + \big|\nabla y_{m_2}'\big|\Big)|\nabla z_m(t)|^2. \qquad (3.16)
\end{aligned}$$

Integrating (3.16) over $(0,t)$ we obtain

$$\begin{aligned}
&|z_m'(t)|^2 + M\Big(|\nabla y_{m_2}(t)|^2\Big)|\nabla z_m(t)|^2 + (2-\eta)\int_0^t |\nabla z_m'(s)|^2\,ds \\
&\quad \le |z_{1m}|^2 + M\Big(|\nabla y_{m_2}(0)|^2\Big)|\nabla z_{0m}|^2 \\
&\qquad + k_4(\eta)\int_0^t \Big(1 + \big|\nabla y_{m_2}'(s)\big|\Big)|\nabla z_m(s)|^2\,ds. \qquad (3.17)
\end{aligned}$$

From the first estimate, using (1.1), choosing $\eta > 0$ sufficiently small and employing Gronwall's lemma we obtain the second estimate

$$\begin{aligned}
&|z_m'(t)|^2 + |\nabla z_m(t)|^2 + \int_0^t |\nabla z_m'(s)|^2 \\
&\quad \le k_5(T)\Big(|z_{1m}|^2 + |\nabla z_{0m}|^2\Big). \qquad (3.18)
\end{aligned}$$

The Third Estimate

Multiplying (3.4) by $g_{jm}''(t)$ and summing over $(0,t)$ we have

$$\begin{aligned}
&|y_m''(t)|^2 + M\Big(|\nabla y_m(t)|^2\Big)\big(\nabla y_m(t), \nabla y_m''(t)\big) + \frac{1}{2}\frac{d}{dt}|\nabla y_m'(t)|^2 \\
&\quad = \big(f(t), y_m''(t)\big) + \frac{d}{dt}\big(g(t), y_m'(t)\big)_{\Gamma_0} - \big(g'(t), y_m'(t)\big)_{\Gamma_0}. \qquad (3.19)
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&M\Big(|\nabla y_m(t)|^2\Big)\big(\nabla y_m(t), \nabla y_m''(t)\big) \\
&\quad = \frac{d}{dt}\Big[M\Big(|\nabla y_m(t)|^2\Big)\big(\nabla y_m(t), \nabla y_m'(t)\big)\Big] \\
&\qquad - 2M'\Big(|\nabla y_m(t)|^2\Big)|\big(\nabla y_m(t), \nabla y_m'(t)\big)|^2 - M\Big(|\nabla y_m(t)|^2\Big)|\nabla y_m'(t)|^2
\end{aligned}$$

and taking into account the first estimate, from (3.19) we obtain

$$\frac{1}{2}|y_m''(t)|^2 + \frac{d}{dt}\Big[M\big(|\nabla y_m(t)|^2\big)\big(\nabla y_m(t), \nabla y_m'(t)\big)\Big] + \frac{1}{2}\frac{d}{dt}|\nabla y_m'(t)|^2$$
$$\leq \frac{1}{2}|f(t)|^2 + C_1|g'(t)|_{\Gamma_0}^2 + C_2|\nabla y_m'(t)|^2 + \frac{d}{dt}\big(g(t), y_m'(t)\big)_{\Gamma_0}, \tag{3.20}$$

where C_1 and C_2 are positive constants.

Integrating (3.20) over $(0, t)$ it follows that

$$\begin{aligned}
&\int_0^t |y_m''(s)|^2\, ds + 2M\big(|\nabla y_m(t)|^2\big)\big(\nabla y_m(t), \nabla y_m'(t)\big) \\
&\quad - 2M\big(|\nabla y_{0m}|^2\big)\big(\nabla y_{0m}, \nabla y_{1m}\big) + |\nabla y_m'(t)|^2 - |\nabla y_{1m}|^2 \\
&\quad \leq \|f\|_{L^2(0,T;\,L^2(\Omega))}^2 + 2C_1\|g'\|_{L^2(0,T;\,L^2(\Gamma_0))}^2 \\
&\qquad + \big(g(t), y_m'(t)\big)_{\Gamma_0} - \big(g(0), y_{1m}\big)_{\Gamma_0} \\
&\qquad + 2C_2\int_0^t |\nabla y_m'(s)|^2\, ds.
\end{aligned} \tag{3.21}$$

Now, from (3.5) and (3.21) there exists a positive constant C_3 such that

$$\int_0^t |y_m''(s)|^2\, ds + 2M\big(|\nabla y_m(t)|^2\big)\big(\nabla y_m(t), \nabla y_m'(t)\big) + |\nabla y_m'(t)|^2$$
$$\leq C_3 + \tfrac{1}{2}|\nabla y_m'(t)|^2 + 2C_2\int_0^t |\nabla y_m'(s)|^2\, ds.$$

Consequently, from the first estimate and from the last inequality we conclude

$$\int_0^t |y_m''(s)|^2\, ds + \tfrac{1}{2}|\nabla y_m'(t)|^2$$
$$\leq C_4 + \tfrac{1}{4}|\nabla y'(t)|^2 + 2C_2\int_0^t |\nabla y_m'(s)|^2\, ds,$$

that is,

$$\int_0^t |y_m''(s)|^2\, ds + \tfrac{1}{4}|\nabla y_m'(t)|^2 \leq C_4 + 2C_2\int_0^t |\nabla y_m'(s)|^2\, ds. \tag{3.22}$$

Finally from (3.22) and using Gronwall's lemma we obtain the third estimate

$$\int_0^t |y_m''(s)|^2\, ds + |\nabla y_m'(t)|^2 \le L_3, \tag{3.23}$$

where L_3 is a positive constant independent of $m \in \mathbf{N}$ and $t \in [0, T]$.

Due to estimates (3.18) and (3.23) we can extract a subsequence (y_μ) of (y_m) such that

$$y_\mu \to y \qquad \text{strongly in } C^0([0,T]; V), \tag{3.24}$$

$$y_\mu' \to y' \qquad \text{strongly in } C^0\big([0,T]; L^2(\Omega)\big), \tag{3.25}$$

$$y_\mu' \to y' \qquad \text{strongly in } L^2(0,T; V), \tag{3.26}$$

$$y_\mu' \rightharpoonup y' \qquad \text{weakly star in } L^\infty(0,T; V), \tag{3.27}$$

$$y_\mu'' \rightharpoonup y'' \qquad \text{weakly in } L^2\big(0,T; L^2(\Omega)\big). \tag{3.28}$$

The above estimates are sufficient to pass to the limit in the linear terms of (3.4).

Analysis of the Nonlinear Term. From (3.24) we have

$$|\nabla y_\mu(t)|^2 \to |\nabla y(t)|^2 \qquad \text{in } C^0[0,T]$$

and as $M \in C^1([0,\infty), \mathbf{R})$ we obtain

$$M\big(|\nabla y_\mu(t)|^2\big) \to M\big(|\nabla y(t)|^2\big) \qquad \text{in } C^0[0,T], \tag{3.29}$$

which is enough to pass to the limit in the nonlinear term.

From the above estimates and using standard arguments we prove that there exists a function $y\colon Q \to \mathbf{R}$ satisfying

$$y'' - M\big(|\nabla y|^2\big)\,\Delta y - \Delta y' = f \qquad \text{in } L^2\big(0,\infty; L^2(\Omega)\big). \tag{3.30}$$

Also, taking into account that

$$-\Delta\big[M\big(|\nabla y|^2\big) y + y'\big] \qquad \text{in } L^2(\Omega)$$

and

$$M\big(|\nabla y|^2\big) y + y' \in V$$

from the generalized Green's formula we infer

$$\frac{\partial}{\partial\nu}\big(M\big(|\nabla y|^2\big)y + y'\big) = g \qquad \text{in } L^2\big(0,\infty; L^2(\Gamma_0)\big). \tag{3.31}$$

Remark 2. We observe that for a.e. $t \geq 0$ the function $y\colon \Omega \to \mathbf{R}$ is the weak solution to the elliptic problem

$$\begin{cases} -\Delta\big[M\big(|\nabla y|^2\big)y + y'\big] = f - y'' \in L^2(\Omega), \\ M(|\nabla y^2)y + y' = 0 \qquad \text{on } \Gamma_1, \\ \dfrac{\partial}{\partial\nu}\big(M\big(|\nabla y|^2\big)y + y'\big) = g \qquad g \in L^2(\Gamma_0). \end{cases}$$

Since $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is empty, the theory of elliptic problems gives

$$y \in L^2\big(0,\infty; V \cap H^{3/2}(\Omega)\big).$$

Now, if we consider $g \in H^1(0,\infty; H^{1/2}(\Gamma_0))$ one has

$$y \in L^2\big(0,\infty; V \cap H^2(\Omega)\big).$$

However, if $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$ unfortunately we cannot use the elliptic regularities mentioned above although we can repeat all the arguments considered. At this point it is convenient to note that if $\Gamma_1 = \emptyset$ our estimates lead us to obtain a solution to problem $(*)$ verifying $y \in C^0(0,\infty; H^1(\Omega)) \cap C^1(0,\infty; L^2(\Omega))$.

Uniqueness. Let y and $\hat{y}$ be two solutions of problem $(*)$. Then, defining $z = y - \hat{y}$ and repeating the same arguments already used in the second estimate we obtain $|\nabla z| = |z'| = 0$.

Solvability of Weak Solutions

We have just proved the existence of solutions to problem $(*)$ when the initial data are smooth. However, when

$$\{y^0, y^1, f, g\} \in V \times L^2(\Omega) \times L^2\big(0,\infty; L^2(\Omega)\big) \times L^2\big(0,\infty; L^2(\Gamma_0)\big)$$

there exist

$$\begin{aligned}\big\{y^0_\mu, y^1 \;\; f_\mu, g_\mu\big\} \in V \cap H^2(\Omega) \times V \cap H^2(\Omega) \times H^1\big(0,\infty; L^2(\Omega)\big) \\ \times H^1\big(0,\infty; L^2(\Gamma_0)\big)\end{aligned}$$

such that

$$\{y_\mu^0, y_\mu^1, f_\mu, g_\mu\} \to \{y^0, y^1, f, g\}$$
$$\text{in } V \times L^2(\Omega) \times L^2\big(0,\infty; L^2(\Omega)\big) \times L^2\big(0,\infty; L^2(\Gamma_0)\big) \tag{3.32}$$

and, by density arguments and using considerations analogous to those used in the first and second estimates, we can find a sequence $\{y_\mu\}$ of solutions to problem $(*)$ such that

$$y_\mu \in C^0([0,T];V), \qquad y'_\mu \in C^0\big([0,T]; L^2(\Omega)\big),$$
$$y''_\mu \in L^2\big(0,T; L^2(\Omega)\big)$$

and

$$y_\mu \to y \qquad \text{strongly in } C^0\big([0,T];V\big), \tag{3.33}$$

$$y'_\mu \to y' \qquad \text{strongly in } C^0\big([0,T]; L^2(\Omega)\big). \tag{3.34}$$

Moreover,

$$y'_\mu \to y' \qquad \text{strongly in } L^2(0,T;V). \tag{3.35}$$

The above converges are sufficient to pass to the limit in order to obtain a weak solution of $(*)$ which satisfies

$$y'' - M\big(|\nabla y|^2\big)\,\Delta y - \Delta y' = f \qquad \text{in } L^2_{\mathrm{loc}}(0,\infty;V'). \tag{3.36}$$

Moreover, we also deduce

$$\frac{\partial}{\partial \nu}\Big(M\big(|\nabla y|^2\big)y + y'\Big) = g \qquad \text{in } L^2_{\mathrm{loc}}\big(0,\infty; L^2(\Gamma_0)\big). \tag{3.37}$$

Indeed, let us consider the following elliptic problems:

$$\begin{cases} -\Delta p = f & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma_0; \end{cases} \tag{3.38}$$

$$\begin{cases} -\Delta q = y' & \text{in } \Omega, \\ q = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial q}{\partial \nu} = g & \text{on } \Gamma_0; \end{cases} \tag{3.39}$$

which admit unique solutions

$$p, q \in L^2_{\text{loc}}(0,\infty,\mathscr{H}), \qquad \text{where } \mathscr{H} = \{u \in V;\ \Delta u \in L^2(\Omega)\}. \tag{3.40}$$

On the other hand, from (3.36) we can write

$$-\Delta\left[M(|\nabla y|^2)y + y'\right] = f - y'' \qquad \text{in } L^2_{\text{loc}}(0,\infty;V')$$

and considering (3.38) and (3.39) one has

$$-\Delta\left[M(|\nabla y|^2)y + y'\right] = -\Delta p + \Delta q' \qquad \text{in } D'_{\text{loc}}(0,\infty;V').$$

Then, we deduce

$$\int_0^T (-\Delta)\left[M(|\nabla y|^2)y + y'\right](t)\theta(t)\,dt$$
$$= \int_0^T (-\Delta)p(t)\theta(t)\,dt + \int_0^T (-\Delta)q(t)\theta'(t)\,dt \qquad \text{in } V'$$

for all $\theta \in D(0,T)$ and consequently

$$\int_0^T \left[M(|\nabla y|^2)y + y'\right](t)\theta(t)\,dt$$
$$= \int_0^T p(t)\theta(t)\,dt + \int_0^T q(t)\theta'(t)\,dt \qquad \text{in } V.$$

The last equality combined with (3.40) allows us to conclude that

$$M(|\nabla y|^2)y + y' = p - q' \qquad \text{in } H^{-1}_{\text{loc}}(0,\infty;\mathscr{H}). \tag{3.41}$$

In the same way, considering for each $\mu \in \mathbf{N}$

$$\begin{cases} -\Delta p_\mu - f_\mu \text{ in } \Omega, \\ p_\mu = 0 \text{ on } \Gamma_1, \\ \dfrac{\partial p_\mu}{\partial \nu} = 0 \text{ on } \Gamma_0, \end{cases} \tag{3.42}$$

and

$$\begin{cases} -\Delta q_\mu = y'_\mu \text{ in } \Omega, \\ q_\mu = 0 \text{ on } \Gamma_1, \\ \dfrac{\partial q_\mu}{\partial \nu} = g_\mu \text{ on } \Gamma_0, \end{cases} \tag{3.43}$$

one has

$$p_\mu, q_\mu \in L^2_{\text{loc}}(0,\infty;\mathscr{H}) \quad \text{and} \quad M\left(|\nabla y_\mu|^2\right) y_\mu + y'_\mu = p_\mu - q'_\mu. \tag{3.44}$$

Next, we are going to prove that

$$q_\mu \to q \qquad \text{in } L^2_{\text{loc}}(0,\infty;\mathscr{H}). \tag{3.45}$$

Indeed, first taking into account the generalized Green's formula and considering (3.39) and (3.43) we infer

$$\int_\Omega |\nabla(q_\mu - q)|^2\, dx = \int_\Omega (y'_\mu - y')(q_\mu - q)\, dx + \int_{\Gamma_0} (g_\mu - g)(q_\mu - q)\, d\Gamma.$$

Making use of the inequality $ab \le (1/4\eta)a^2 + b^2$, $\eta > 0$, considering the Cauchy–Schwarz inequality and integrating over $[0, T]$, we obtain

$$\begin{aligned} &\int_0^T |\nabla q_\mu(t) - \nabla q(t)|^2\, dt \\ &\qquad \le C \int_0^T \left\{ \left| y'_\mu(t) - y'(t) \right|^2 + |g_\mu(t) - g(t)|^2_{\Gamma_0} \right\} dt. \end{aligned} \tag{3.46}$$

But from (3.39) and (3.43) we have

$$\begin{aligned} &\|q_\mu - q\|^2_{L^2(0,T;\mathscr{H})} \\ &\quad = \int_0^T |\nabla q_\mu(t) - \nabla q(t)|^2\, dt + \int_0^T |\Delta q_\mu(t) - \Delta q(t)|^2\, dt \\ &\quad = \int_0^T |\nabla q_\mu(t) - \nabla q(t)|^2\, dt + \int_0^T \left| y'_\mu(t) - y'(t) \right|^2 dt. \end{aligned} \tag{3.47}$$

Combining (3.46)–(3.47) and the convergences (3.32) and (3.34) we conclude (3.45). Analogously we deduce

$$p_\mu \to p \qquad \text{in } L^2_{\text{loc}}(0,\infty;\mathscr{H}). \tag{3.48}$$

Then, from (3.45) and (3.48) it follows that

$$p_\mu - q'_\mu \to p - q' \qquad \text{in } H^{-1}_{\text{loc}}(0,\infty;\mathscr{H})$$

and therefore, from the above convergence and considering (3.31) and (3.44), we conclude

$$g_\mu = \frac{\partial}{\partial\nu}(p_\mu - q'_\mu) \to \frac{\partial}{\partial\nu}(p - q') \qquad \text{in } H^{-1}_{\text{loc}}\big(0,\infty; H^{-1/2}(\Gamma_0)\big). \tag{3.49}$$

But, in view of (3.32) one has

$$g_\mu \to g \qquad \text{in } L^2_{\text{loc}}\big(0,\infty; L^2(\Gamma_0)\big). \tag{3.50}$$

Then, combining (3.49) and (3.50) we deduce the desired in (3.37).

Uniqueness

Let y_1 and y_2 be weak solutions to problem $(*)$. Then, defining $z = y_1 - y_2$ one has

$$\begin{cases} z'' - \Delta\Big\{M\big(|\nabla y_1|^2\big)y_1 + y'_1 \\ \qquad -\Big(M\big(|\nabla y_2|^2\big)y_2 + y'_2\Big)\Big\} = 0 \qquad \text{in } L^2_{\text{loc}}(0,\infty;V') \\ z = 0 \qquad \text{on } \Sigma_0 \\ \dfrac{\partial}{\partial\nu}\Big\{M\big(|\nabla y_1|^2\big)y_1 + y'_1 \\ \qquad -\Big(M\big(|\nabla y_2|^2\big)y_2 + y'_2\Big)\Big\} = 0 \qquad \text{in } L^2_{\text{loc}}\big(0,\infty; L^2(\Gamma_0)\big) \\ z(0) = 0, \qquad z'(0) = 0. \end{cases}$$

Then, noting that $z' \in L^2(0,\infty;V)$ the duality $\langle z'', z'\rangle_{V'\times V}$ makes sense and consequently we conclude

$$\frac{1}{2}\frac{d}{dt}|z'(t)|^2 + |\nabla z'(t)|^2 = M\big(|\nabla y_2|^2\big)\big(\nabla y_2(t), \nabla z'(t)\big) - M\big(|\nabla y_1|^2\big)\big(\nabla y_1(t), \nabla z'(t)\big).$$

From the equality above and making use of analogous arguments considered in the second estimate we deduce that $|z'(t)|^2 = |\nabla z(t)|^2 = 0$. This concludes the proof.

4. ASYMPTOTIC BEHAVIOUR

In this section we are going to obtain the uniform decay for strong solutions of $(*)$. Using density arguments we conclude the same for weak solutions.

The derivative of the energy defined in (1.3) is given by

$$E'(t) = -|\nabla y'(t)|^2 + (f(t), y'(t)) + (g(t), y'(t))_{\Gamma_0}. \tag{4.1}$$

We define the perturbed energy by

$$E_\varepsilon(t) = E(t) + \varepsilon\psi(t), \tag{4.2}$$

where

$$\psi(t) = \int_\Omega y' y\, dx. \tag{4.3}$$

Let $\lambda > 0$ be a constant such that

$$|v| \leq \lambda |\nabla v|, \qquad \forall v \in V. \tag{4.4}$$

PROPOSITION 4.1. *There exists a positive constant C_1 such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1 E(t), \qquad \forall t \geq 0 \text{ and } \forall \varepsilon > 0.$$

Proof. From (1.1), (4.3), and (4.4) we have

$$|\psi(t)| \leq \lambda\lambda_0^{-1/2}|y'(t)|\left[\hat{M}\left(|\nabla y(t)|^2\right)\right]^{1/2} \leq \lambda\lambda_0^{-1/2}E(t). \tag{4.5}$$

From (4.2) and (4.5) we can write

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon C_1 E(t),$$

where $C_1 = \lambda\lambda_0^{-1/2}$. This concludes the proof. ∎

PROPOSITION 4.2. *There exist C_2 and $C_3 = C_3(\varepsilon)$ and ε_1 positive constants such that*

$$E'_\varepsilon(t) \leq -\varepsilon C_2 E(t) + C_3\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right),$$
$$\forall t \geq 0 \text{ and } \forall \varepsilon \in (0, \varepsilon_1].$$

Proof. Taking the derivative of $\psi(t)$ defined in (4.3) and using

$$y'' = M\left(|\nabla y(t)|^2\right)\Delta y + \Delta y' + f$$

it follows that

$$\psi'(t) = M\big(|\nabla y(t)|^2\big)(\Delta y(t), y(t)) + (\Delta y(t), y(t)) \\ + |y'(t)|^2 + (f(t), y(t)). \tag{4.6}$$

From the generalized Green's formula and taking into account that

$$\frac{\partial}{\partial \nu}\Big(M\big(|\nabla y(t)|^2\big)y + y'\Big) = g \qquad \text{on } \Sigma_0$$

from (4.6) we obtain

$$\psi'(t) = -M\big(|\nabla y(t)|^2\big)|\nabla y(t)|^2 - (\nabla y(t), \nabla y'(t)) + (g(t), y(t))_{\Gamma_0} \\ + (f(t), y(t)) + |y'(t)|^2. \tag{4.7}$$

On the other hand we note that $|\nabla y(t)| \le K$, $\forall t \ge 0$ and therefore defining

$$k_1 = \min\{M(\lambda);\ \lambda \in [0, K]\} \quad \text{and} \quad k_2 = \max\{M(\lambda);\ \lambda \in [0, K]\}$$

we have

$$\hat{M}\big(|\nabla y(t)|^2\big) \le \frac{k_2}{k_1} M\big(|\nabla y(t)|^2\big)|\nabla y(t)|^2. \tag{4.8}$$

Subtracting and adding the term $|y'(t)|^2$ in the equality (4.7) and taking (4.8) into account we get

$$\psi'(t) \le -LE(t) - (\nabla y(t), \nabla y'(t)) + (g(t), y(t))_{\Gamma_0} \\ + (f(t), y(t)) + 2|y'(t)|^2, \tag{4.9}$$

where $L = \min\{1, k_1/k_2\}$.

Now, from (1.1), (4.4), and (4.9) for an arbitrary $\eta > 0$ we obtain

$$\psi'(t) \le -\big(L - 6\eta\lambda_0^{-1}\big)E(t) + \frac{1}{4\eta}|\nabla y'(t)|^2 + \frac{C_0^2}{4\eta}|g(t)|_{\Gamma_0}^2 \\ + \frac{\lambda}{4\eta}|f(t)|^2 + 2\lambda^2|\nabla y'(t)|^2, \tag{4.10}$$

where C_0 is a positive constant such that $|v|_{\Gamma_0} \le C_0|\nabla v|$, $\forall v \in V$.

Choosing $\eta = L\lambda_0/12$ from (4.10) it follows that

$$\psi'(t) \le -\frac{L}{2}E(t) + \left(\frac{3\lambda_0^{-1}}{L} + 2\lambda^2\right)|\nabla y'(t)|^2 + \frac{3\lambda^2}{L\lambda_0}|f(t)|^2 + \frac{2C_0^2}{L\lambda_0}|g(t)|_{\Gamma_0}^2. \tag{4.11}$$

From (4.1), (4.2), and (4.11) we can write

$$\begin{aligned} E'_\varepsilon(t) &= E'(t) + \varepsilon\psi'(t) \\ &\le -(1-\varepsilon N)|\nabla y'(t)|^2 - \varepsilon\frac{L}{2}E(t) \\ &\quad + \left(\frac{1}{2\varepsilon} + \frac{3\lambda^2}{L\lambda_0}\right)|f(t)|^2 + \left(\frac{1}{2\varepsilon} + \varepsilon\frac{2C_0^2}{L\lambda_0}\right)|g(t)|_{\Gamma_0}^2, \end{aligned} \tag{4.12}$$

where

$$N = \frac{2\lambda_0^{-1}}{L} + \frac{5}{2}\lambda^2 + \frac{1}{2}C_0^2.$$

Defining

$$\varepsilon_1 = \frac{1}{N}$$

and considering $\varepsilon \in (0, \varepsilon_1]$ from (4.12) we conclude

$$E'_\varepsilon(t) \le -\varepsilon C_2 E(t) + C_3(\varepsilon)\left(|f(t)|^2 + |g(t)|_{\Gamma_0}^2\right).$$

This finishes the proof. ∎

Proof of the Exponential Decay. We define

$$\varepsilon_0 = \min\left\{\frac{1}{2C_1}, \varepsilon_1\right\}.$$

From Proposition 4.1 we have

$$(1 - C_1\varepsilon)E(t) \le E_\varepsilon(t) \le (1 + C_1\varepsilon)E(t). \tag{4.13}$$

Since $\varepsilon \le 1/2C_1$, then

$$\tfrac{1}{2}E(t) \le E_\varepsilon(t) \le \tfrac{3}{2}E(t) \le 2E(t), \qquad \forall t \ge 0 \tag{4.14}$$

and therefore

$$-\varepsilon C_2 E(t) \leq -\frac{\varepsilon}{2} C_2 E_\varepsilon(t). \tag{4.15}$$

Hence, from (4.15) and considering Proposition 4.2 we obtain

$$E'_\varepsilon(t) \leq -\frac{\varepsilon}{2} C_2 E_\varepsilon(t) + C_2\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right).$$

Consequently,

$$\frac{d}{dt}\left(E_\varepsilon(t)\exp\left(\frac{\varepsilon}{2}C_2 t\right)\right) \leq C_3\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right)\exp\left(\frac{\varepsilon}{2}C_2 t\right).$$

Integrating the above inequality over $[0, t]$ we get

$$\begin{aligned} E_\varepsilon(t) \leq{}& \exp\left(-\frac{\varepsilon}{2}C_2 t\right)E_\varepsilon(0) \\ &+ C_3 \exp\left(-\frac{\varepsilon}{2}C_2 t\right)\int_0^t \exp\left(\frac{\varepsilon}{2}C_2 s\right)\left(|f(s)|^2 + |g(s)|^2_{\Gamma_0}\right) \end{aligned}$$

and taking into consideration (4.14) we see that

$$E(t) \leq \left(3E(0) + 2C_3\int_0^t \exp\left(\frac{\varepsilon}{2}C_2 s\right)\left(|f(t)|^2 + |g(t)|^2_{\Gamma_0}\right)\right)\exp\left(-\frac{\varepsilon}{2}C_2 t\right). \tag{4.16}$$

Combining (4.16) with assumption (2.3) we prove the desired decay and finish the proof of Theorem 2.1. ∎

Further Remarks

(1) Let $-\Delta$ be the operator defined by the triple $\{V, L^2(\Omega), a(u,v)\}$, where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u, v \in V,$$

and

$$D(-\Delta) = \left\{u \in V \cap H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\}.$$

We recall that the spectral theorem for self-adjoint operators guarantees the existence of a complete orthonormal system (ω_ν) of $L^2(\Omega)$ given by the eigenfunctions of $-\Delta$. If (λ_ν) are the eigenvalues of $-\Delta$, then $\lambda_\nu \to +\infty$ as $\nu \to +\infty$. Now, since $-\Delta$ is positive, given $\alpha > 0$ one has

$$D\left[(-\Delta)^{\alpha}\right] = \left\{u \in L^2(\Omega);\ \sum_{\nu=1}^{\infty} \lambda_\nu^{2\alpha} |(u, \omega_\nu)|^2 < \infty\right\}$$

and

$$(-\Delta)^{\alpha} u = \sum_{\nu=1}^{\infty} \lambda_\nu^{\alpha} (u, \omega_\nu)\, \omega_\nu, \qquad \text{for all } u \in D\left[(-\Delta)^{\alpha}\right].$$

In $D[(-\Delta)^\alpha]$ we consider the topology given by

$$\|u\|_{D[-(\Delta)^{\alpha}]} = |(-\Delta)^{\alpha} u|_{L^2(\Omega)}.$$

We observe that such operators are self-adjoints, that is,

$$\left((-\Delta)^{\alpha} u, v\right) = \left(u, (-\Delta)^{\alpha} v\right), \qquad \text{for all } u, v \in D\left[(-\Delta)^{\alpha}\right],$$

and, moreover, $D[(-\Delta)^{1/2}] = V$.

Choosing smooth initial data, considering a basis in $D[(-\Delta)^\alpha]$ for $\alpha \geq 1$ and noting that the injection $D[(-\Delta)^{\alpha/2}] \hookrightarrow D[(-\Delta)^{1/2}]$ is continuous we can repeat all the considerations used in the above estimates in order to extend our results for a general class of damped problems

$$\begin{cases} y_{tt} - M\left(\int_\Omega |\nabla y|^2\, dx\right) \Delta y + (-\Delta)^{\alpha} y_t = f & \text{in } Q, \\ y = 0 \quad \text{on } \Sigma_1, & \\ M\left(\int_\Omega |\nabla y|^2\, dx\right) \dfrac{\partial y}{\partial \nu} + \dfrac{\partial}{\partial \nu}\left[(-\Delta)^{\alpha - 1} y_t\right] = g & \text{on } \Sigma_0, \\ y(0) = y^0, \qquad y_t(0) = y^1, \qquad \text{in } \Omega. & \end{cases}$$

(2) We observe that we can relax assumption (1.1) considering $M(\lambda) \geq 0$, $\forall \lambda \geq 0$. Indeed, for this end it is sufficient to consider more than one estimate taking $w = y_m(t)$ in (3.4).

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